MAXIMUM SOLUTIONS OF NORMALIZED RICCI FLOWS ON 4-MANIFOLDS

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ABSTRACT. We consider maximum solution $g(t)$, $t \in [0, +\infty)$, to the normalized Ricci flow. Among other things, we prove that, if (M, ω) is a smooth compact symplectic 4-manifold such that $b_2^+(M) > 1$ and let $g(t), t \in [0, \infty)$, be a solution to (1.3) on M whose Ricci curvature satisfies that $|\text{Ric}(g(t))| \leq 3$ and additionally $\chi(M) = 3\tau(M) > 0$, then there exists an $m \in \mathbb{N}$, and a sequence of points $\{x_{j,k} \in M\}$, $j = 1, \cdots, m$, satisfying that, by passing to a subsequence,

$$(M, g(t_k + t), x_{1,k}, \cdots, x_{m,k}) \xrightarrow{d_{GH}} (\coprod_{j=1}^{m} N_j, g_\infty, x_{1,\infty}, \cdots, x_{m,\infty}),$$

$t \in [0, \infty)$, in the m-pointed Gromov-Hausdorff sense for any sequence $t_k \longrightarrow \infty$, where (N_j, g_∞), $j = 1, \cdots, m$, are complete complex hyperbolic orbifolds of complex dimension 2 with at most finitely many isolated orbifold points. Moreover, the convergence is C^∞ in the non-singular part of $\coprod_1^m N_j$ and $\text{Vol}_{g_0}(M) = \sum_{j=1}^{m} \text{Vol}_{g_\infty}(N_j)$, where $\chi(M)$ (resp. $\tau(M)$) is the Euler characteristic (resp. signature) of M.

1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold. The Perelman λ-functional

$$(1.1) \qquad \lambda_M(g) = \inf_{f \in C^\infty(M)} \{\mathcal{F}(g, f) : \int_M e^{-f} d\text{vol}_g = 1\}$$

where $\mathcal{F}(g, f) = \int_M (R_g + |\nabla f|^2) e^{-f} d\text{vol}_g$ and R_g is the scalar curvature of g. Note that $\lambda_M(g)$ is the lowest eigenvalue of the operator $-4\triangle + R_g$. By [Pe1] the gradient flow of the Perelman λ-functional is the Hamilton's the Ricci-flow evolution equation

$$(1.2) \qquad \frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t))$$

The normalized Ricci flow equation on an n-manifold M reads

$$(1.3) \qquad \frac{\partial}{\partial t} g(t) = -2Ric(g(t)) + \frac{2\overline{R}}{n} g(t)$$

where Ric (resp. $\overline{R}$) denotes the Ricci tensor (resp. the average scalar curvature $\frac{\int_M R dv}{\int_M dv}$). Note that (1.2) and (1.3) differ only by a change of scale in space and time, and the volume $\text{Vol}(g(t))$ is constant in t. If $\dim M = n$, $\overline{\lambda}_M(g) = \lambda_M(g)\text{Vol}_g(M)^{\frac{2}{n}}$ is invariant up to rescaling the metric. Perelman [Pe1] has proved that $\overline{\lambda}_M(g(t))$ is non-decreasing along the Ricci flow $g(t)$ whenever $\overline{\lambda}_M(g(t)) \leq 0$. This leads to the

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Perelman invariant $\overline{\lambda}_M$ by taking supremum of $\overline{\lambda}_M(g)$ in the set of all Riemannian metrics on M.

By [AIL] the Perelman invariant $\overline{\lambda}_M$ is equal to the Yamabe invariant whenever $\overline{\lambda}_M \leq 0$, after the earlier estimations (cf. [An5] [Pe2] [Le4] [FZ] and [Kot]). In particular, if (M, g) is a smooth compact oriented 4-manifold with a Spin^c-structure $\mathfrak{c}$ which is a monopole class (i.e., the associated Seiberg-Witten equation possesses an irreducible solution) so that that $c_1^2(\mathfrak{c})[M] > 0$, by [FZ] $\overline{\lambda}_M \leq -\sqrt{32\pi^2 c_1^2(\mathfrak{c})[M]}$. Moreover, g is a Kähler-Einstein metric of negative scalar curvature if and only if $\overline{\lambda}_M(g) = -\sqrt{32\pi^2 c_1^2(\mathfrak{c})[M]}$. However, there are plenty of 4-manifolds where the Perelman invariant $\overline{\lambda}_M = -\sqrt{32\pi^2 c_1^2(\mathfrak{c})[M]}$ but do not admit any Kähler Einstein metric. It is natural to study 4-manifolds with these extremal property. For such a 4-manifold M, to seek for an "optimal" Riemannian metric on M with respect to the Perelman functional $\overline{\lambda}_M : \mathcal{M} \to \mathbb{R}$, we want to consider a *maximal solution* $g(t)$ which is a solution of the Ricci flow (1.3). We call a longtime solution $g(t)$, $t \in [0, +\infty)$, to the Ricci flow (1.3) a *maximum solution* if $\lim_{t \to \infty} \overline{\lambda}_M(g(t)) = \overline{\lambda}_M$. For a compact 3-manifold, by Perelman [Pe2] all solutions of the Ricci flow (1.2) with surgery exist for longtime and are maximum solutions, provided $\overline{\lambda}_M \leq 0$. In the paper [FZZ] obstructions are found for the longtime solutions with bounded curvature to (1.3).

In this paper we are going to study the maximum solutions of (1.3) with bounded Ricci curvatures instead. To avoid technique terminology we only state our results for symplectic 4-manifolds by using the celebrated work of Taubes [Ta]: if (M, ω) is a compact symplectic manifold with $b_2^+(M) > 1$ (the dimension of self-dual harmonic 2-forms of M), the spin^c-structure induced by ω is a monopole class. Moreover, in this situation $c_1^2(\mathfrak{c})[M] = 2\chi(M) + 3\tau(M)$, where $\chi(M)$ (resp. $\tau(M)$) is the Euler characteristic (resp. signature) of M.

Theorem 1.1. *Let (M, ω) be a smooth compact symplectic 4-manifold satisfying that $b_2^+(M) > 1$ and $2\chi(M) + 3\tau(M) > 0$. If $g(t), t \in [0, \infty)$, is a solution to (1.3) such that $|Ric(g(t))| \leq 3$, and*

$$\lim_{t \to \infty} \overline{\lambda}_M(g(t)) = -\sqrt{32\pi^2(2\chi(M) + 3\tau(M))},$$

then there exists an $m \in \mathbb{N}$, and sequences of points $\{x_{j,k} \in M\}$, $j = 1, \cdots, m$, satisfying that, by passing to a subsequence,

$$(M, g(t_k + t), x_{1,k}, \cdots, x_{m,k}) \xrightarrow{d_{GH}} (\coprod_{j=1}^{m} N_j, g_\infty, x_{1,\infty}, \cdots, , x_{m,\infty}),$$

$t \in [0, \infty)$, in the m-pointed Gromov-Hausdorff sense for any sequence $t_k \longrightarrow \infty$, where (N_j, g_∞), $j = 1, \cdots, m$, are complete Kähler-Einstein orbifolds of complex dimension 2 with at most finitely many isolated orbifold points. The scalar curvature (resp. volume) of g_∞ is

$$-Vol_{g_0}(M)^{-\frac{1}{2}}\sqrt{32\pi^2(2\chi(M) + 3\tau(M))} \quad (\text{resp. } Vol_{g_0}(M) = \sum_{j=1}^{m} Vol_{g_\infty}(N_j))$$

Moreover, the convergence is C^∞ in the non-singular part of $\coprod_1^m N_j$.

We first remark that, if the diameters $\text{diam}_{g(t_k)}(M)$ possess a uniform upper bound, then $m = 1$, and N_1 is a compact Kähler-Einstein orbifold. Secondly, if the Ricci curvature bound in the above theorem is replaced by a uniform bound of sectional curvature, then every (N_j, g_∞), $j = 1, \cdots, m$ are complete Kähler-Einstein manifolds. By the same arguments as in [An5][An6], $\coprod_{j=1}^{m} N_j$ can weakly embed in M, $\coprod_{j=1}^{m} N_j \subset\subset M$, i.e. for any compact subset $K \subset \coprod_{j=1}^{m} N_j$, there is a smooth embedding $F_K : K \longrightarrow M$. Furthermore, there exists a sufficiently large compact subset $K \subset \coprod_{j=1}^{m} N_j$ such that $M \backslash K$ admits an F-structure of positive rank. This type geometric decomposition seems very useful to understand the diffeomorphism type of 4-manifolds.

Theorem 1.2. *Let (M, ω) be a smooth compact symplectic 4-manifold such that $b_2^+(M) > 1$ and let $g(t), t \in [0, \infty)$, be a solution to (1.3) such that $|R(g(t))| \leq 12$. If in addition $\chi(M) = 3\tau(M) > 0$, then*

$$\lim_{t \to \infty} \overline{\lambda}_M(g(t)) = -\sqrt{32\pi^2(2\chi(M) + 3\tau(M))}$$

Moreover, if $|Ric(g(t))| \leq 3$, the Kähler-Einstein metric g_∞ in Theorem 1.1 is complex hyperbolic.

To conclude the section we point out that the main result in Theorem 1.1 (resp. Corollary 1.2) holds if the manifold is not symplectic but a compact oriented 4-manifold with a monopole class c_1 (i.e. with a spin^c-structure with non-vanishing Seiberg-Witten invariant) so that $c_1^2 = 2\chi(M) + 3\tau(M) > 0$.

2. Preliminaries

2.1. **Monopole class.** Let (M, g) be a compact oriented Riemannian 4-manifold with a Spin^c structure $\mathfrak{c}$. Let $b_2^+(M)$ denote the dimension of the space of self-dual harmonic 2-forms in M. Let $S_\mathfrak{c}^\pm$ denote the Spin^c-bundles associated to $\mathfrak{c}$, and let L be the determinant line bundle of $\mathfrak{c}$. There is a well-defined Dirac operator

$$\mathcal{D}_A : \Gamma(S_\mathfrak{c}^+) \longrightarrow \Gamma(S_\mathfrak{c}^-)$$

Let $c : \wedge^* T^*M \longrightarrow \text{End}(S_\mathfrak{c}^+ \oplus S_\mathfrak{c}^-)$ denote the Clifford multiplication on the Spin^c-bundles, and, for any $\phi \in \Gamma(S_\mathfrak{c}^\pm)$, let

$$q(\phi) = \overline{\phi} \otimes \phi - \frac{1}{2}|\phi|^2\text{id}.$$

The Seiberg-Witten equations read

$$(2.1) \qquad \begin{aligned} \mathcal{D}_A\phi &= 0 \\ c(F_A^+) &= q(\phi) \end{aligned}$$

where A is an Hermitian connection on L, and F_A^+ is the self-dual part of the curvature of A.

A solution of (2.1) is called *reducible* if $\phi \equiv 0$; otherwise, it is called *irreducible*. If (ϕ, A) is a resolution of (2.1), one calculates easily that

$$(2.2) \qquad |F_A^+| = \frac{1}{2\sqrt{2}}|\phi|^2,$$

The Bochner formula reads

(2.3) $$0 = -2\triangle|\phi|^2 + 4|\nabla^A\phi|^2 + R_g|\phi|^2 + |\phi|^4,$$

where R_g is the scalar curvature of g.

The Seiberg-Witten invariant can be defined by counting the irreducible solutions of the Seiberg-Witten equations (cf. [Le2]).

Definition 2.2. ([K1]) Let M be a smooth compact oriented 4-manifold. An element $\alpha \in H^2(M,\mathbb{Z})/\text{torsion}$ is called a monopole class of M if and only if there exists a Spin^c-structure $\mathfrak{c}$ on M with first Chern class $c_1 \equiv \alpha(\text{mod torsion})$, *so that the Seiberg-Witten equations have a solution for every Riemannian metric g on M.*

By the celebrated work of Taubes [Ta], if (M,ω) is a compact symplectic 4-manifold with $b_2^+(M) > 1$, the canonical class of (M,ω) is a monopole class.

2.3. **Kato's inequality.** Let (M,g) be a Riemannian Spin^c-manifold of dimension n, the following Kato inequality is useful.

Proposition 2.4. (Proposition 2.2 in [BD]) *Let ϕ be a harmonic* Spin^c*-spinor on (M,g), i.e. $\mathcal{D}_A\phi = 0$, where $\mathcal{D}_A$ is the Dirac operator and A is an Hermitian connection on the determinant line bundle. Then*

(2.4) $$|\nabla|\phi||^2 \leq \frac{n-1}{n}|\nabla^A\phi|^2 \leq |\nabla^A\phi|^2$$

at all points where ϕ is non-zero. Moreover, $|\nabla|\phi||^2 = |\nabla^A\phi|^2$ occurs only if $\nabla^A\phi \equiv 0$.

Note that the arguments in the proof of Proposition 2.2 in [BD] can be used to prove this proposition without any change, where the same conclusion was derived for Spin-spinor ϕ. For any $\epsilon > 0$, let $|\phi|_\epsilon^2 = |\phi|^2 + \epsilon^2$. If ϕ is harmonic, by above proposition,

(2.5) $$|\nabla|\phi|_\epsilon|^2 \leq \frac{|\phi|}{|\phi|_\epsilon}|\nabla|\phi||^2 \leq \frac{n-1}{n}|\nabla^A\phi|^2 \leq |\nabla^A\phi|^2$$

at points where $\phi(p) \neq 0$. Since $\{p \in M : \phi(p) \neq 0\}$ is dense in M for harmonic ϕ, we conclude that (2.5) holds everywhere in M.

2.5. **Chern-Gauss-Bonnet formula and Hirzebruch signature formula.** Let (M,g) be a compact closed oriented Riemannian 4-manifold, $\chi(M)$ and $\tau(M)$ are the Euler number and the signature of M respectively. The Chern-Gauss-Bonnet formula and the Hirzebruch signature theorem say that

(2.6) $$\chi(M) = \frac{1}{8\pi^2}\int_M(\frac{R_g^2}{24} + |W_g|^2 - \frac{1}{2}|Ric^o|^2)dv_g, \quad \text{and}$$

(2.7) $$\tau(M) = \frac{1}{12\pi^2}\int_M(|W_g^+|^2 - |W_g^-|^2)dv_g,$$

where $Ric^o = Ric(g) - \frac{R_g}{4}g$ is the Einstein tensor, W_g^+ and W_g^- are the self-dual and anti-self-dual Weyl tensors respectively (cf. [B]). If g is a Kähler-Einstein metric, then

(2.8) $$R_g^2 = 24|W_g^+|^2,$$

(cf. [Le3]) which will be used in the proof of Theorem 1.1.

By Chern-Gauss-Bonnet formula, one has an L^2-bound of the curvature operator $Rm(g)$ by the bounds of Ricci curvature, i.e. if $|Ric(g)| < C$, then

$$(2.9) \qquad \int_M |Rm(g)|^2 dv_g \leq 8\pi^2 \chi(M) + C_1 Vol_g(M),$$

where C and C_1 are constants independent of (M, g).

Let (N, g) be a complete Ricci-flat Einstein 4-manifold. Assume that

$$(2.10) \qquad \int_N |Rm(g)|^2 dv_g < \infty, \quad \text{and} \quad \text{Vol}_g(B_g(x, r)) \geq Cr^4,$$

for all $r > 0$, a point $x \in N$, and a positive constant C. By Theorem 2.11 of [N], (N, g) is ALE. (i.e, Asymptotically Locally Euclidean space) of order 4. It is well-known that N is asymptotic to the cone on the spherical space form S^3/Γ, where $\Gamma \subset SO(4)$ is a finite group. The Chern-Gauss-Bonnet formula implies that

$$(2.11) \qquad \chi(N) = \frac{1}{8\pi^2} \int_N |Rm(g)|^2 dv_g + \frac{1}{|\Gamma|}$$

(cf. [N] and [An1]).

2.6. **Curvature estimates for 4-manifolds.** Now let's recall a result of [CT], which is important to the proof of Theorem 1.1. Let (M, g) be a complete Riemannian 4-manifold. A subset $U \subset M$ such that for all $p \in U$, $\sup\limits_{B_g(p,1)} Ric(g) \geq -3$, is called ϱ-collapsed if for all $p \in U$,

$$Vol_g(B_g(p, 1)) \leq \varrho.$$

By Theorem 0.1 in [CG], there is a constant ε_4 such that if U is ϱ-collapsed with sectional curvature $|K_g| \leq 1$ and $\varrho \leq \varepsilon_4$, then U carries an F-structure of positive rank.

Theorem 2.7. (Remark 5.11 and Theorem 1.26 in [CT]) *There exist constants $\delta > 0$, $c > 0$ such that: if (M, g) is a complete Riemannian 4-manifold with $|Ric(g)| \leq 3$ and*

$$\int_M |Rm(g)|^2 dv_g \leq C,$$

and if $E \subset M$ is a bounded open subset such that $T_1(E) = \{x \in M : dist(x, E) \leq 1\}$ is ε_4-collapsed with

$$\int_{B_g(x,1)} |Rm(g)|^2 dv_g \leq \delta \quad (for \quad all \quad T_1(E)),$$

then

$$\int_E |Rm(g)|^2 dv_g \leq cVol_g(A_{0,1}(E)),$$

where $A_{0,1}(E) = T_1(E) \backslash E$.

3. The limiting behavior of Ricci flow

In this section we study the limiting behavior of Ricci-flow with bounded Ricci curvatures on 4-manifolds. We will assume in this section that M is a smooth closed oriented 4-manifold with $\overline{\lambda}_M < 0$, and $g(t)$, $t \in [0, +\infty)$, is a longtime solution of the normalized Ricci flow (1.3) with bounded Ricci-curvature. By normalization we may assume that $|Ric(g(t))| \leq 3$. By (2.9) there is a constant C independent of t such that

$$\int_M |Rm(g(t))|^2 dv_{g(t)} \leq C.$$

Let us denoted by V the volume $\mathrm{Vol}_{g(0)}(M) = \mathrm{Vol}_{g(t)}(M)$, and $\breve{R}(g(t)) = \min_{x \in M} R(g(t))(x)$ the minimum of the scalar curvature of $g(t)$. It is easy to see that $\breve{R}(g(t)) \leq \overline{\lambda}_M V^{-\frac{1}{2}} < 0$.

Lemma 3.1. *(3.1.1)* $\lim_{t \to \infty} \lambda_M(g(t)) = \lim_{t \to \infty} \overline{R}(g(t)) = \lim_{t \to \infty} \breve{R}(g(t)) = \overline{R}_\infty$
(3.1.2) $\lim_{t \to \infty} \int_M |R(g(t)) - \overline{R}(g(t))| dv_{g(t)} = 0,$
(3.1.3) $\lim_{t \to \infty} \int_M |Ric^o(g(t))|^2 dv_{g(t)} = 0.$

Proof. By Perelman [Pe1] $\lambda_M(g(t))$ is a non-decreasing function on t, therefore the limit $\lim_{t \to \infty} \lambda_M(g(t))$ exists since $\overline{\lambda}_M < 0$. Now let us denote by $\overline{R}_\infty$ the limit $\lim_{t \to \infty} \lambda_M(g(t))$. Note that $\overline{R}_\infty \leq \overline{\lambda}_M V^{-\frac{1}{2}} < 0$. To prove (3.1.1), we first prove that both $\lim_{t \to \infty} \overline{R}(g(t))$ and $\lim_{t \to \infty} \breve{R}(g(t))$ exist and take values $\overline{R}_\infty$. By the same arguments as in the proof of Proposition 2.6 and Lemma 2.7 of [FZZ] we get that

$$\lim_{t \to \infty} \overline{R}(g(t)) - \breve{R}(g(t)) = 0.$$

Observe that $\overline{R}(g(t)) \geq \lambda_M(g(t)) \geq \breve{R}(g(t))$ (cf. [KL] (92.3)). Therefore $\lim_{t \to \infty} \overline{R}(g(t)) = \overline{R}_\infty = \lim_{t \to \infty} \breve{R}(g(t))$. This proves (3.1.1).

Note that

$$\int_M |R(g(t)) - \overline{R}(g(t))| dv_{g(t)} \leq \int_M (R(g(t)) - \breve{R}(g(t))) dv_{g(t)} + \int_M (\overline{R}(g(t)) - \breve{R}(g(t))) dv_{g(t)}$$
$$= 2(\overline{R}(g(t)) - \breve{R}(g(t)))V$$

(3.1.2) follows from (3.1.1).

By Lemma 3.1 in [FZZ],

$$\int_0^\infty \int_M |Ric^o(g(t))|^2 dv_{g(t)} dt < \infty,$$

and, by Lemma 1 in [Ye], we have

$$\frac{d}{dt} \int_M |Ric^o(g(t))|^2 dv_{g(t)} \leq -2 \int_M |\nabla Ric^o(g(t))|^2 dv_{g(t)} + 4 \int_M |Rm||Ric^o(g(t))|^2 dv_{g(t)} < D,$$

where D is a constant independent of t. By the same argument as in the proof of Proposition 2.6 in [FZZ] (3.1.3) follows. $\square$

The following is the main result of this section, which is an analogy of Theorem 10.5 in [CT], where the same conclusion was derived for closed oriented Einstein 4-manifolds with the same negative Einstein constant. The key point in our case is to use Lemma 3.1 to get non-collapsing balls and to prove the limiting metric is an Einstein metric (cf. Lemma 3.3 and Lemma 3.4 below).

Proposition 3.2. *Let M be a smooth closed oriented 4-manifold with $\overline{\lambda}_M < 0$. If $g(t), t \in [0, \infty)$ is a solution to (1.3) such that $|Ric(g(t))| \leq 3$, and $\{t_k\}$ is a sequence of times tends to infinity such that*

$$diam_{g_k}(M) \longrightarrow \infty,$$

when $k \longrightarrow \infty$, where $g_k = g(t_k)$, then there exists an $m \in \mathbb{N}$, and sequences of points $\{x_{j,k} \in M\}$, $j = 1, \cdots, m$, satisfying that, by passing to a subsequence,

$$(M, g_k, x_{1,k}, \cdots, x_{m,k}) \xrightarrow{d_{GH}} (\coprod_{j=1}^{m} N_j, g_\infty, x_{1,\infty}, \cdots, , x_{m,\infty})$$

in the m-pointed Gromov-Hausdorff sense for $k \to \infty$, where (N_j, g_∞) $j = 1, \cdots, m$ are complete Einstein 4-orbifolds with at most finitely many isolated orbifold points $\{q_i\}$. The scalar curvature (resp. volume) of g_∞ is

$$\overline{R}_\infty = \lim_{t \longrightarrow \infty} \lambda_M(g(t)), \qquad (resp. \quad V = Vol_{g_0}(M) = \sum_{j=1}^{m} Vol_{g_\infty}(N_j)).$$

Furthermore, in the regular part of N_j, $\{g_k\}$ converges to g_∞ in both $L^{2,p}$ (resp. $C^{1,\alpha}$) sense for all $p < \infty$ (resp. $\alpha < 1$).

We divide the proof of Proposition 3.2 into several useful lemmas.

A key result in the paper [CT] shows that, for any compact oriented Einstein 4-manifold (X, g) with Einstein constant -3, there exists a constant C depending only on the Euler number of X, and a point $x \in X$ such that $Vol_g(B_g(x, 1)) \geq CVol_g(X)$ (cf. Theorem 0.14 [CT]). Cheeger-Tian remarked that the same result continues to hold for 4-manifolds which are sufficiently negatively Ricci pinched. The following lemmas is an analogy of the result for the metric g_k in Proposition 3.2.

Lemma 3.3. *There exists a constant $v > 0$, and a sequence $\{x_k\} \subset M$ such that*

$$Vol_{g_k}(B_{g_k}(x_k, 1)) \geq v.$$

Proof. Let $\varepsilon_4 > 0$ be the critical constant of Cheeger-Tian (cf. §1 [CT]), i.e., if X is a Riemannian 4-manifold which is ε_4-collapsed with locally bounded curvature, then X carries an F-structure of positive rank. We may assume that, for all $x \in M$ and g_k, $Vol_{g_k}(B_{g_k}(x, 1)) < \varepsilon_4$. By a standard covering argument, for any k, there exist finitely many points $q_1, \cdots, q_l$ such that $E = M \backslash \bigcup_{i=1}^{l} B_{g_k}(q_i, 1)$ satisfies the hypothesis of Theorem 2.7. Moreover, $l \leq C\delta^{-1}$ where C and δ are the constants in Theorem 2.7. Therefore, by Theorem 2.7 we conclude that, there is a constant C_1 independent of k such that

$$(3.1) \qquad \int_E |R(g_k)|^2 dv_k \leq 6 \int_E |Rm(g_k)|^2 dv_k \leq C_1 \sum_{i=1}^{l} Vol_{g_k}(B_{g_k}(q_i, 1)).$$

On the other hand, by Lemma (3.1.2)

$$|\int_E (R(g_k)^2 - \overline{R}(g_k)^2)dv_k| \le 24 \int_E |R(g_k) - \overline{R}(g_k)|dv_k \overset{k \to \infty}{\longrightarrow} 0.$$

Therefore

(3.2)

$$
\begin{aligned}
\frac{1}{2}\overline{R}_\infty^2 \mathrm{Vol}_{g_k}(E) - \int_E R(g_k)^2 dv_k &\le \overline{R}(g_k)^2 \mathrm{Vol}_{g_k}(E) - \int_E R(g_k)^2 dv_k \\
&= \int_E (\overline{R}(g_k)^2 - R(g_k)^2)dv_k \\
&\le \frac{1}{4}\overline{R}_\infty^2 V
\end{aligned}
$$

for sufficiently large k since $\overline{R}_\infty \le \overline{\lambda}_M V^{-\frac{1}{2}} < 0$. By inserting (3.1) we get that

$$
\begin{aligned}
\frac{1}{2}\overline{R}_\infty^2(V - \sum_{i=1}^l Vol_{g_k}(B_{g_k}(q_i,1))) - \frac{1}{4}\overline{R}_\infty^2 V &\le \frac{1}{2}\overline{R}_\infty^2 \mathrm{Vol}_{g_k}(E) - \frac{1}{4}\overline{R}_\infty^2 V \\
&\le C_1 \sum_{i=1}^l \mathrm{Vol}_{g_k}(B_{g_k}(q_i,1)),
\end{aligned}
$$

and

$$V \le C_2 \sum_{i=1}^l \mathrm{Vol}_{g_k}(B_{g_k}(q_i,1)),$$

where C_2 is a constant independent of k. Therefore, there is at least a ball among the l balls whose volume is at least $\frac{V}{C_2 l}$. The desired result follows. $\square$

Assuming that $\mathrm{diam}_{g_k}(M) \to \infty$ for $k \to \infty$, by using the technique developed in [An3], the analogue of Theorem 3.3 in [An2] holds (cf. Theorem 2.3 in [An4]), i.e. there exist a sequence of points $\{x_k\} \subset M$ such that, by passing to a subsequence,

$$\{(M, g_k, x_k)\} \overset{d_{GH}}{\longrightarrow} (N_\infty, g_\infty, x_\infty)$$

where N_∞ is a 4-orbifold with only isolated orbifold points $\{q_i\}$, g_∞ is a complete C^0 orbifold metric, and g_∞ is a $C^{1,\alpha} \cap L^{2,p}$ Riemannian metric on the regular part of N_∞, for all $p < \infty$ and $\alpha < 1$. Furthermore, $\{g_k\}$ converges to g_∞ in the $L^{2,p}$ (resp. $C^{1,\alpha}$) sense on the regular part of N_∞, i.e. for any $r \gg 1$ and k, there is a smooth embedding $F_{k,r} : B_{g_\infty}(x_\infty, r)\backslash \bigcup_i B_{g_\infty}(q_i, r^{-1}) \subset N_\infty \to M$ such that, by passing to a subsequence, $F_{k,r}^* g_k$ converge to g_∞ in both $L^{2,p}$ and $C^{1,\alpha}$ senses.

Lemma 3.4. g_∞ *is an Einstein orbifold metric with scalar curvature* $\overline{R}_\infty$.

Proof. We first prove that g_∞ is an Einstein metric with scalar curvature $\overline{R}_\infty$ on the regular part of N_∞. Since $F_{k,r}^* g_k$ converge to g_∞ in the $L^{2,p}$(resp. $C^{1,\alpha}$) sense on $B_{g_\infty}(p_\infty, r)\backslash \bigcup_i B_{g_\infty}(q_i, r^{-1})$, for any r, by Lemma 3.1, we obtain that

$$0 \le \int_{B_{g_\infty}(p_\infty,r)\backslash \bigcup_i B_{g_\infty}(q_i,r^{-1})} |Ric^o(g_\infty)|^2 dv_\infty \le \lim_{k \to \infty} \int_M |Ric^o(g_k)|^2 dv_k = 0,$$

$$0 \leq \int_{B_{g_\infty}(p_\infty, r) \backslash \bigcup_i B_{g_\infty}(q_i, r^{-1})} |R(g_\infty) - \overline{R}_\infty| dv_\infty \leq \lim_{k \to \infty} \int_M |R(g_k) - \overline{R}(g_k)| dv_k = 0.$$

Therefore g_∞ is a $C^{1,\alpha}$ Riemannian metric on $B_{g_\infty}(p_\infty, r) \backslash \bigcup_i B_{g_\infty}(q_i, r^{-1})$ which satisfies the Einstein equation in the weak sense. By elliptic regularity theory, g_∞ is a smooth Einstein metric with scalar curvature $\overline{R}_\infty$.

Since g_∞ is a C^0-orbifold metric, i.e. for any orbifold point $q_i \in N_\infty$, there is a neighborhood $U_i \cong B(0,r)/\Gamma$ of q_i such that $\widetilde{g}_\infty$ is a C^0-Riemannian metric on $B(0,r) \subset \mathbb{R}^4$ where $\Gamma \subset SO(4)$ is a finite subgroup acting freely on S^3, and $\widetilde{g}_\infty|_{B(0,r)\backslash\{0\}}$ is the pull-back metric of g_∞. Note that $\widetilde{g}_\infty$ is a smooth Einstein metric on $B(0,r)\backslash\{0\}$ satisfying that $\int_{B(0,r)} |Rm(\widetilde{g}_\infty)|^2 dv_{\widetilde{g}_\infty} < C < \infty$. By the arguments as in [An1] and [Ti], $\widetilde{g}_\infty$ is a C^∞ Einstein metric on $B(0,r)$ (cf. the proof of Theorem C in [An1], and Section 4 in [Ti]). Hence g_∞ is an Einstein orbifold metric. $\square$

By the discussion before Lemma 3.4 we may choose ℓ sequences of points $\{x_{j,k}\} \subset M$, $j = 1, \cdots, \ell$, such that $\text{dist}_{g_k}(x_{i,k}, x_{j,k}) \overset{k \to \infty}{\longrightarrow} \infty$ for any $i \neq j$, and

$$(3.3) \qquad \{(M, g_k, x_{1,k}, \cdots, x_{\ell,k})\} \overset{d_{GH}}{\longrightarrow} (\coprod_{j=1}^{\ell} N_j, g_\infty, x_{1,\infty}, \cdots, x_{\ell,\infty})$$

where $(N_j, g_\infty, x_{j,\infty})$, $j = 1, \cdots, \ell$ are complete Einstein 4-orbifolds with only isolated singular points and scalar curvatures $\overline{R}_\infty$. Furthermore, $\{g_k\}$ converges to g_∞ in both $L^{2,p}$ (resp. $C^{1,\alpha}$) sense on the regular parts of N_j, $j = 1, \cdots, \ell$. Note that

$$(3.4) \qquad V \geq \sum_{i=1}^{\ell} \text{Vol}_{g_\infty}(N_j).$$

Lemma 3.5. *The number of orbifold points of $\coprod_{j=1}^{\ell} N_j$ is less than a constant depending only on the Euler characteristic $\chi(M)$.*

Proof. For each orbifold point $q \in N_j$, there exist a sequence $\{q_k\} \subset M$, and two constants $r \gg r_1 > 0$ such that:

(3.5.1) $q \in B_{g_\infty}(x_{j,\infty}, r)$;

(3.5.2) $B_{g_\infty}(q, r_1) \backslash B_{g_\infty}(q, \sigma)$ lies in the regular part of $B_{g_\infty}(x_{j,\infty}, r)$ for any $\sigma < r_1$;

(3.5.3) $(B_{g_k}(q_k, r_1) \backslash B_{g_k}(q_k, \sigma), g_k) \overset{C^{1,\alpha}}{\longrightarrow} (B_{g_\infty}(q, r_1) \backslash B_{g_\infty}(q, \sigma), g_\infty)$.

By the definition of harmonic radius (cf. [An3]), the harmonic radii of all points in $B_{g_k}(q_k, r_1) \backslash B_{g_k}(q_k, \sigma)$ have a uniform lower bound, saying $\mu > 0$, a constant depending on σ but independent of k.

Clearly, there is a positive constant v_0 (e.g., $\frac{1}{2}\text{Vol}_{g_\infty}(B_{g_\infty}(x_{j,\infty}, r))$) such that $\text{Vol}_{g_k}(B_{g_k}(x_{j,k}, r)) \geq v_0$. Note that the Sobolev constants $C_{S,k}$ of $B_{g_k}(x_{j,k}, r)$ are bounded from below by a constant depending only on v_0, r (cf. [An2] and [Cr]). Therefore, by [An2] again we get that $\text{Vol}_{g_k}(B_{g_k}(q_k, s)) \geq Cs^4$ for any $s \ll 1$, where C is independent of k.

Let us denote by $r_{h,k}$ the infimum of the harmonic radii of g_k in the ball $B_{g_k}(q_k, r_1)$. Note that $r_{h,k} \xrightarrow{k \to \infty} 0$ since q is a orbifold point (cf. [An3]). Therefore, there is a point $\bar{q}_k \in B_{g_k}(q_k, \sigma)$ so that $r_h(\bar{q}_k) = r_{h,k}$ for sufficiently large k.

Consider the normalized balls $(B_{g_k}(q_k, r_1), r_{h,k}^{-2} g_k)$, which have harmonic radii at least 1. By passing to a subsequence if necessary,

$$(B_{g_k}(q_k, r_1), r_{h,k}^{-2} g_k, \bar{q}_k) \xrightarrow{C^{1,\alpha}} (W, \bar{g}_\infty, \bar{q})$$

where $(W, \bar{g}_\infty)$ is a complete Ricci-flat 4-manifold satisfying that

(3.5) $$\mathrm{Vol}_{\bar{g}_\infty}(B_{\bar{g}_\infty}(\bar{q}, r)) \geq Cr^4$$

for any $r > 0$. It is obvious that

$$\int_W |Rm(\bar{g}_\infty)|^2 dv_{\bar{g}_\infty} \leq \liminf_{k \to \infty} \int_M |Rm(g_k)|^2 dv_k \leq C.$$

Therefore $(W, \bar{g}_\infty)$ is an Asymptotically Locally Euclidean space (cf. Theorem 2.11 in [N] or [An1]), which is asymptotic to a cone of S^3/Γ where $\Gamma \subset SO(4)$ is a finite group acting freely on S^3. By the Chern-Gauss-Bonnet formula

(3.6) $$\chi(W) = \frac{1}{8\pi^2} \int_W |Rm(\bar{g}_\infty)|^2 dv_{\bar{g}_\infty} + \frac{1}{|\Gamma|}.$$

By [An1] W is isometric to $\mathbb{R}^4$, provided $|\Gamma| = 1$. Since the harmonic radius of $\bar{g}_\infty$ at $\bar{q}$ is 1, hence $\bar{g}_\infty$ can not be the Euclidean metric. Hence $|\Gamma| \geq 2$. It is easy to verify that $\chi(W) \geq 1$. By (3.6) we get that

$$\int_W |Rm(\bar{g}_\infty)|^2 dv_{\bar{g}_\infty} \geq 4\pi^2.$$

This proves that every orbifold point contributes to $\liminf_{k \to \infty} \int_M |Rm(g_k)|^2 dv_k$ at least $4\pi^2$. By the rescaling invariance of the integral we conclude that the number of orbifold points $\beta \leq \frac{C}{4\pi^2}$. $\square$

The following lemma is an analogue of a result in Cheeger-Tian [CT].

Lemma 3.6. $\ell < \chi(M) + \beta + 1$, where $\beta := \#\{$number of orbifold points in Lemma 3.5$\}$.

Proof. Suppose not, i.e, $\ell \geq \chi(M) + \beta + 1$, by definition there are at least $\chi(M) + 1$ components of $\coprod_1^\ell N_j$ which are smooth complete non-compact Einstein 4-manifolds of finite volume, for simplicity saying $N_1, \cdots, N_s$, where $s \geq \chi(M) + 1$. By Theorem 4.5 in [CT], for each $1 \leq j \leq s$,

$$\int_{N_j} |Rm(g_\infty)|^2 dv_{g_\infty} \geq 8\pi^2.$$

Since $(M, g_k, x_{k,j}) \xrightarrow{L^{2,p}} (N_j, g_\infty, x_{\infty,j})$, by Chern-Gauss-Bonnet formula and (3.1.3) in Lemma 3.1 we get that

$$8\pi^2 \chi(M) = \lim_{k \to \infty} \int_M |Rm(g_k)|^2 dv_{g_k} \geq \sum \int_{N_j} |Rm(g_\infty)|^2 dv_{g_\infty} \geq 8\pi^2 (\chi(M) + 1).$$

A contradiction. $\square$

Let m denote the maximal value of all possible choice of the base point sequences in (3.3), which has a upper bound by Lemma 3.6.

Lemma 3.7. *Let* $M_{k,r} = M \backslash \bigcup_{j=1}^{m} B_{g_k}(x_{j,k}, r)$. *For sufficiently large* r, *there is a constant* C *independent of* r *such that*

$$(3.7) \qquad \lim_{k \to \infty} Vol_{g_k}(M_{k,r}) \leq C \sum_{j=1}^{m} Vol_{g_\infty}(N_j \backslash B_{g_\infty}(x_{j,\infty}, \frac{r}{2})),$$

$$(3.8) \qquad \sum_{j=1}^{m} Vol_{g_\infty}(N_j) = V.$$

Proof. We may choose $r \gg 1$ such that, for any $y \in \coprod_{j=1}^{m}(N_j \backslash B_{g_\infty}(x_{j,\infty}, r-1))$, $\mathrm{Vol}_{g_\infty}(\mathrm{B}_{g_\infty}(y,1)) \leq \frac{1}{2}\varepsilon_4$, where $\varepsilon_4 > 0$ is the critical constant of Cheeger-Tian (cf. proof of Lemma 3.3 or §1 [CT]).

Now we claim that there is a constant $k_0 \gg 1$ such that, for any $k > k_0$ and any $x \in M_{k,r}$, $\mathrm{Vol}_{g_k}(B_{g_k}(x,1)) \leq \varepsilon_4$.

If it is false, without loss of generality we may assume a sequence of points $\{y_k\} \subset M_{k,r}$ such that

$$(3.9) \qquad \mathrm{Vol}_{g_k}(\mathrm{B}_{g_k}(y_k, 1)) > \varepsilon_4$$

Observe that the distance $\mathrm{dist}_{g_k}(y_k, x_{j,k}) \to \infty$ as $k \to \infty$ for all $1 \leq j \leq m$. Otherwise, assuming $\mathrm{dist}_{g_k}(y_k, x_{j,k}) < \rho$ for some j and $\rho > 0$, we get that $F_{j,k,\rho}^{-1}(y_k) \to y_\infty \in B_{g_\infty}(x_{j,\infty}, \rho) \backslash B_{g_\infty}(x_{j,\infty}, r-1)$, and so

$$(3.10) \qquad \mathrm{Vol}_{g_k}(\mathrm{B}_{g_k}(y_k, 1)) \to \mathrm{Vol}_{g_\infty}(\mathrm{B}_{g_\infty}(y_\infty, 1)) \leq \frac{1}{2}\varepsilon_4$$

when $k \to \infty$, since $F_{j,k,\rho}^* g_k$ $C^{1,\alpha}$-converges to $g_{j,\infty}$, where

$$(3.11) \qquad F_{j,k,\rho} : B_{g_\infty}(x_{j,\infty}, \rho) \backslash \bigcup_i B_{g_\infty}(q_i, \rho^{-1}) \subset N_\infty \to M$$

is a smooth embedding so that $F_{j,k,\rho}^* g_k$ converges to g_∞ in the $C^{1,\alpha}$-sense (cf. the discussion before Lemma 3.4). A contradiction to (3.9).

Note that $(M, g_k, y_k) \xrightarrow{d_{GH}} (N_\infty, g_\infty, y_\infty)$ where N_∞ is a complete 4-orbifold different from each of N_j, $1 \leq j \leq m$. This violates the choice of maximality of m. Hence we have proved the claim.

By a standard covering argument, for any k, there exist finitely many points $z_{1,k}, \cdots, z_{I,k}$ such that $E_{k,r} = M_{k,r} \backslash \bigcup_{i=1}^{I} B_{z_{i,k}}(1)$ satisfies the hypothesis of Theorem 2.7, where I is independent of k. By Theorem 2.7, there is a constant C independent of k such that

$$\int_{E_{k,r}} |R(g_k)|^2 dv_k \leq 6 \int_{E_{k,r}} |Rm(g_k)|^2 dv_k \leq C(\sum_{i=1}^{I} \mathrm{Vol}_{g_k}(B_{g_k}(z_{i,k}, 1)) + \mathrm{Vol}_{g_k}(A_{0,1}(M_{k,r}))).$$

By Lemma 3.1, for $k \gg 1$, we have

$$(3.12) \qquad \int_{E_{k,r}} |R(g_k) - \overline{R}(g_k)| dv_k < \int_M |R(g_k) - \overline{R}(g_k)| dv_k \longrightarrow 0.$$

By (3.2) we get

$$\frac{1}{2}\overline{R}_\infty^2 \mathrm{Vol}_{g_k}(E_{k,r}) - \int_{E_{k,r}} R(g_k)^2 dv_k \ \leq \ \int_{E_{k,r}} (R(g_k)^2 - \overline{R}(g_k)^2) dv_k$$

$$\leq \ 24 \int_{E_{k,r}} |R(g_k) - \overline{R}(g_k)| dv_k,$$

Since $\mathrm{Vol}_{g_k}(E_{k,r}) \geq \mathrm{Vol}_{g_k}(M_{k,r}) - \sum_{i=1}^{I} \mathrm{Vol}_{g_k}(B_{g_k}(z_{i,k}, 1))$, by the above together we get immediately that
(3.13)

$$\mathrm{Vol}_{g_k}(M_{k,r}) \leq C\Big(\sum_{i=1}^{I} \mathrm{Vol}_{g_k}(B_{g_k}(z_{i,k}, 1)) + \mathrm{Vol}_{g_k}(A_{0,1}(M_{k,r}))\Big) + 24 \int_{E_{k,r}} |R(g_k) - \overline{R}(g_k)| dv_k.$$

If $\mathrm{dist}_{g_k}(z_{i,k}, x_{j,k}) \to \infty$ for all $1 \leq j \leq m$, by the same argument as above we get that

$$\mathrm{Vol}_{g_k}(B_{g_k}(z_{i,k}, 1)) \to 0$$

when $k \to \infty$. Otherwise, there exists a subsequence $k_s \to \infty$ and an index j such that

$$\mathrm{dist}_{g_{k_s}}(z_{i,k_s}, x_{j,k_s}) < \rho$$

for some constant ρ. In both cases, we obtain

$$\limsup_{k \to \infty} \mathrm{Vol}_{g_k}(B_{g_k}(z_{i,k_s}, 1)) \leq \sum_{j=1}^{m} \mathrm{Vol}_{g_\infty}(N_j \backslash B_{g_\infty}(x_{j,\infty}, \frac{r}{2}))$$

for $r \gg \rho$. Therefore, by (3.12) and (3.13) we conclude immediately (3.7).

By (3.7) it follows that $\lim_{k,r \to \infty} \mathrm{Vol}_{g_k}(M_{k,r}) \to 0$. Hence (3.8) follows. $\qquad \square$

By now Proposition 3.2 follows by the above lemmas.

4. Smooth convergence on the regular part

The main result of this section is the following:

Proposition 4.1. *Let M be a closed 4-manifold satisfying that $\bar{\lambda}_M < 0$ and let $g(t), t \in [0, \infty)$, be a solution to the normalized Ricci flow equation (1.3) on M with uniformly bounded Ricci curvature. If $(M, g(t_k), p_k) \xrightarrow{d_{GH}} (N_\infty, g_\infty, p_\infty)$, where $t_k \to \infty$ and N_∞ is a 4-dimensional orbifold, and $g(t_k) \xrightarrow{C^{1,\alpha}} g_\infty$ on the regular part $\mathcal{R}$ of N_∞ (the compliment of the orbifold points), then, by passing to a subsequence, for all $t \in [0, \infty)$, $(M, g(t_k + t), p_k) \xrightarrow{d_{GH}} (N_\infty, g_\infty(t), p_\infty)$, where $g_\infty(t)$ is a family of smooth metrics on $\mathcal{R}$ solving the normalized Ricci flow equation on $\mathcal{R}$ with $g_\infty(0) = g_\infty$. Moreover, the convergence is smooth on $\mathcal{R} \times [0, \infty)$.*

In [Se] the convergence of Kähler-Ricci flow on compact Kähler manifolds with bounded Ricci curvature was studied. It seems that the arguments in [Se] could be applied to prove Proposition 4.1, but the authors can not follow completely her line. Therefore, we give a quite different approach, where we first give a curvature estimate of the Ricci flow similar to Perelman's pseudolocality theorem. Using this curvature

estimation we prove the limit Ricci flow exists on $\mathcal{R} \times [0, \infty)$. Finally, we prove that $\mathcal{R}$ is exactly the regular part of every subsequence limit of $(M, g(t_k + t), p_k)$, for all $t \in [0, \infty)$. It deserves to point out that our approach works only in dimension 4.

We now give a curvature estimate for the Ricci flow which is an analogy of Perelman's pseudolocality theorem (cf. [Pe1] Thm. 10.1). The difference is that here we use the hypothesis of local almost Euclidean volume growth, instead of the almost Euclidean isoperimetric estimate. The proof is much easier than that of Perelman's pseudolocality theorem.

Theorem 4.2. *There exist universal constants $\delta_0, \epsilon_0 > 0$ with the following property. Let $g(t), t \in [0, (\epsilon_P r_0)^2]$, be a solution to the Ricci flow equation (1.2) on a closed n-manifold M and $x_0 \in M$ be a point. If the scalar curvature*

$$R(x, t) \geq -r_0^{-2} \text{ whenever } \text{dist}_{g(t)}(x_0, x) \leq r_0,$$

and the volume

$$\text{Vol}_{g(t)}(B_{g(t)}(x, r)) \geq (1 - \delta_0)\text{Vol}(B(r)) \text{ for all } B_{g(t)}(x, r) \subset B_{g(t)}(x_0, r_0),$$

where $B(r)$ denotes a ball of radius r in the n-Euclidean space and $\text{Vol}(B(r))$ denotes its Euclidean volume, then the Riemannian curvature tensor satisfies

$$|Rm|_{g(t)}(x, t) \leq t^{-1}, \text{ whenever } \text{dist}_{g(t)}(x_0, x) < \epsilon_0 r_0, \text{ and } 0 < t \leq (\epsilon_0 r_0)^2.$$

In particular, $|Rm|_{g(t)}(x_0, t) \leq t^{-1}$ for all time $t \in (0, (\epsilon_0 r_0)^2]$.

Proof. We use Claim 1 and Claim 2 of Theorem 10.1 in [Pe1] and adopt a contradiction argument. For any given small constants $\epsilon, \delta > 0$, set $\epsilon_0 = \epsilon, \delta_0 = \delta$, then there is a solution to the Ricci flow equation (1.2), say $(M, g(t))$, not satisfying the conclusion of the theorem. After a rescaling, we may assume that $r_0 = 1$. Denote by $\bar{M}$ the non-empty set of pairs (x, t) such that $|Rm|_{g(t)}(x, t) > t^{-1}$, then as in Claim 1 and Claim 2 of Theorem 10.1 in [Pe1], we can choose another space time point $(\bar{x}, \bar{t}) \in \bar{M}$ with $0 < \bar{t} \leq \epsilon^2, \text{dist}_{g(\bar{t})}(x_0, \bar{x}) < \frac{1}{10}$, such that $|Rm|_{g(t)}(x, t) \leq 4Q$ whenever

$$\bar{t} - \frac{1}{2n}Q^{-1} \leq t \leq \bar{t}, \qquad \text{dist}_{g(\bar{t})}(\bar{x}, x) \leq \frac{1}{10}(100n\epsilon)^{-1}Q^{-1/2},$$

where $Q = |Rm|_{g(\bar{t})}(\bar{x}, \bar{t})$. It is remarkable that from the proof of Claim 2 of Theorem 10.1 in [Pe1], each such a space time point (x, t) satisfies

$$\text{dist}_{g(t)}(x, x_0) < \text{dist}_{g(\bar{t})}(x_0, \bar{x}) + (100n\epsilon)^{-1}Q^{-1/2} < \frac{1}{10} + (100n)^{-1} < \frac{1}{2}.$$

Now choosing sequences of positive numbers $\epsilon_k \to 0$ and $\delta_k \to 0$, we obtain a sequence of solutions $(M_k, g_k(t)), t \in [0, \epsilon_k^2]$ and a sequence of points $x_{0,k}, \bar{x}_k \in M_k$ and times $\bar{t}_k$, with each satisfying the assumptions of the theorem and the properties described above. In particular, we have that $Q_k = |Rm_k|_{g_k(\bar{t}_k)}(\bar{x}_k, \bar{t}_k) \to \infty$. Consider the sequence of pointed Ricci flow solutions

$$(B_{g_k(\bar{t}_k)}(\bar{x}_k, \frac{1}{10}(100n\epsilon_k)^{-1}Q_k^{-1/2}), Q_k g_k(Q_k^{-1}t + \bar{t}_k), \bar{x}_k), t \in [-\frac{1}{2n}, 0].$$

Using Hamilton's compactness theorem for solutions to the Ricci flow, we can extract a subsequence which converge to a complete Ricci flow solution $(M_\infty, g_\infty(t), \bar{x}_\infty), t \in (-\frac{1}{2n}, 0]$, with $|Rm_\infty|_{g_\infty(0)}(\bar{x}_\infty, 0) = 1$.

By assumption, the balls

$$B_{g_k(\bar{t}_k)}(\bar{x}_k, \frac{1}{10}(100n\epsilon_k)^{-1}Q_k^{-1/2}) \subset B_{g_k(t)}(x_{0,k}, \frac{1}{2})$$

for any $t \in [\bar{t} - \frac{1}{2n}Q^{-1}, \bar{t}]$, so the scalar curvature $R_k(x,t) \geq -1$ for $t \in [\bar{t} - \frac{1}{2n}Q^{-1}, \bar{t}]$ and $x \in B_{g_k(\bar{t}_k)}(\bar{x}_k, \frac{1}{10}(100n\epsilon_k)^{-1}Q_k^{-1/2})$ and $\mathrm{Vol}_{g_k(t)}(B_{g_k(t)}(x,r)) \geq (1-\delta_k)\mathrm{Vol}(B(r))$ for any metric ball $B_{g_k(t)}(x,r) \subset B_{g_k(\bar{t}_k)}(\bar{x}_k, \frac{1}{10}(100n\epsilon_k)^{-1}Q_k^{-1/2})$, $t \in [\bar{t} - \frac{1}{2n}Q^{-1}, \bar{t}]$. Passing to the limit, we see that $g_\infty(t)$ has scalar curvature $R_\infty \geq 0$ everywhere and local volume $\mathrm{Vol}_{g_\infty(t)}(B_{g_\infty(t)}(z,r)) \geq \mathrm{Vol}(B(r))$ for any balls $B_{g_\infty(t)}(z,r)$ at time $t \in (-\frac{1}{2n}, 0]$. Then the local variation formula of volume implies that $R_\infty \equiv 0$ on $M_\infty \times (-\frac{1}{2n}, 0]$, see [STW] for details. By the evolution of the scalar curvature $\frac{\partial}{\partial t}R_\infty = \triangle R_\infty + 2|Ric_\infty|^2$, we get that $Ric_\infty \equiv 0$ over $M_\infty \times (-\frac{1}{2n}, 0]$. Then the Bishop-Gromov volume comparison theorem implies that $g_\infty(t)$ are flat solutions to the Ricci flow, which contradicts the fact that $|Rm_\infty|(\bar{x}_\infty, 0) = 1$. This ends the proof of the theorem. $\qquad\square$

The next lemma provides a comparison of the curvature of the normalized and un-normalized Ricci flow. By assumption, there is $\bar{C} < \infty$ such that $|Ric| \leq \bar{C}$ everywhere along the flow $(M, g(t))$. Note that by Lemma 3.1, there is some time $T < \infty$ such that $2\overline{R}_\infty \leq \overline{R}(g(t)) \leq \frac{1}{2}\overline{R}_\infty < 0$ whenever $t > T$. Fix any such a time $\bar{t} > T$ and let $h(t)$ and $\tilde{h}(\tilde{t})$ be the solutions to the normalized and unnormalized Ricci flow with initial metric $h(0) = \tilde{h}(0) = g(\bar{t})$ respectively, where $\tilde{t} = \tilde{t}(t)$ is the corresponding rescaled time for t. Denote by $Rm_{\bar{t}}, Ric_{\bar{t}}, R_{\bar{t}}$ and $\widetilde{Rm}_{\bar{t}}, \widetilde{Ric}_{\bar{t}}, \tilde{R}_{\bar{t}}$ the corresponding Riemannian curvature, Ricci curvature and scalar curvature of them, where $|Ric_{\bar{t}}| \leq \bar{C}$ since $h(t) = g(\bar{t}+t)$. Then we have

Lemma 4.3. *The solution $\tilde{h}(\tilde{t})$ exists for all time $\tilde{t} \in [0, \infty)$. Furthermore, there exist constants C and τ depending on $\bar{\lambda}_M$ and $\bar{C}$, such that*

$$t \leq \tilde{t} \leq Ct, |\widetilde{Rm}_{\bar{t}}|(x,\tilde{t}) \leq |Rm_{\bar{t}}|(x,t) \leq C|\widetilde{Rm}_{\bar{t}}|(x,\tilde{t}), \text{ whenever } t \leq \tau.$$

Proof. The solution $h(t)$ has average scalar curvature $\overline{R}(\bar{t}+t) \leq \frac{1}{2}\overline{R}_\infty < 0$, so $\tilde{h}(\tilde{t})$ also has average scalar curvature $\widetilde{\overline{R}} < 0$. From the evolution $\frac{d}{d\tilde{t}}\ln \mathrm{Vol}(\tilde{h}(\tilde{t})) = -\widetilde{\overline{R}}$, the volume $\mathrm{Vol}(\tilde{h}(\tilde{t}))$ increases strictly in $\tilde{t}$, so to normalize it, we need to compress the space and time. Thus $\tilde{t} \geq t$ and $|\widetilde{Rm}_{\bar{t}}|(x,\tilde{t}) \leq |Rm_{\bar{t}}|(x,t)$ for all (x,t). So $\tilde{h}(\tilde{t})$ exists for all time.

The last assertion means that the scaling factor from normalized Ricci flow to the unnormalized one is less than C on the time interval $[0, \tau]$. Consider the evolution of average scalar curvature $\widetilde{\overline{R}}(\tilde{t})$:

$$\frac{d}{d\tilde{t}}\widetilde{\overline{R}} = \frac{\int_M(2|\widetilde{Ric}_{\bar{t}}|^2 - \tilde{R}_{\bar{t}}^2)dv_k}{Vol_{\tilde{h}(\tilde{t})}(M)} + \widetilde{\overline{R}}^2 \leq \Lambda,$$

for some constant $\Lambda = \Lambda(\bar{C})$, since $|\widetilde{Ric}_{\bar{t}}| \leq |Ric_{\bar{t}}| \leq \bar{C}, |\tilde{R}_{\bar{t}}| \leq |R_{\bar{t}}| \leq \bar{C}, |\widetilde{\overline{R}}| \leq |\overline{R}| \leq \bar{C}$. Note that the initial value $\widetilde{\overline{R}}(0) = \overline{R}(g(\bar{t})) \leq \frac{1}{2}\overline{R}_\infty$, so there is some constant $\tilde{\tau} = \tilde{\tau}(\Lambda)$ such that $\widetilde{\overline{R}}(\tilde{t}) \leq \frac{1}{4}\overline{R}_\infty$ for $\tilde{t} \in [0, \tilde{\tau}]$. Thus the scaling factor from normalized Ricci

flow to the unnormalized one, which equals $\frac{\overline{R}(h(t))}{\tilde{\overline{R}}(\tilde{t})}$, is less than 8 on the time interval $\tilde{t} \in [0, \tilde{\tau}]$. Now the result follows, by setting $\tau = \frac{\tilde{\tau}}{8}$ and $C = 8$. $\square$

The following lemma gives the estimation of the local volume along the Ricci flow. As in [Se], the proof uses Theorem A 1.5 of [CC]. By assumption, we have a solution $(M, g(t))$ to the normalized Ricci flow (1.3) and a sequence of times $t_k \to \infty$ and points p_k such that $(M, g(t_k), p_k) \xrightarrow{d_{GH}} (N_\infty, g_\infty, p_\infty)$ with $g(t_k) \xrightarrow{C^{1,\alpha}} g_\infty$ on the regular part $\mathcal{R}$ of the orbifold N_∞. For the space M or N_∞, let $\mathcal{R}_{\epsilon, \rho}$ be the set of points x such that $d_{GH}(B(x, r), B(r)) < \epsilon r$ for any $r \le u$, where $u \ge \rho$ is some constant depending on x. Here and after, $B(r)$ denotes a ball of radius r in 4-Euclidean space and $B(x, r)$ the metric ball of radius r with center x in a metric space. A weak version is $\mathcal{WR}_{\epsilon, \rho}$, the set of points x such that there is $u \ge \rho$ with $d_{GH}(B(x, u), B(u)) < \epsilon u$.

Lemma 4.4. *For each $q \in \mathcal{R}$, choose a sequence $q_k \in M$ that converge to q. Then for any $\epsilon > 0$, there exist $k_0, \eta, \rho > 0$ such that*

$$\mathrm{Vol}(B_{g(t_k + t)}(q'_k, r)) \ge (1 - \epsilon)\mathrm{Vol}(B(r)), \forall r < \rho, k_0 < k,$$

whenever $B_{g(t_k + t)}(q'_k, r) \subset B_{g(t_k)}(q_k, \rho)$ and $t \in [-\eta, \eta]$.

Proof. By the boundedness of Ricci tensor, there is a universal constant $\Lambda = \Lambda(\bar{C}) > 1$ such that $B_{g(t)}(p, \Lambda^{-1}r) \subset B_{g(s)}(p, r) \subset B_{g(t)}(p, \Lambda r)$ for all $t, s \in [t_k - 1, t_k + 1], p \in M$ and $r > 0$. By Theorem A.1.5 of [CC], for fixed $\epsilon > 0$, there are $\delta = \delta(\epsilon, n), \rho = \rho(\epsilon, n) > 0$ such that $x \in \mathcal{WR}_{\delta, r}$ implies $\mathrm{Vol}(B_{g(t)}(x, r)) \ge (1 - \epsilon)\mathrm{Vol}(B(r))$ for each $r \le \rho$ and $x \in M$. So by definition, it suffice to show $q'_k \in \mathcal{R}_{\delta, \rho}$ with respect to each metric $g(t), t \in [t_k - \eta, t_k + \eta]$, whenever $q'_k \in B_{g(t_k)}(q_k, \Lambda\rho)$, for some constant $\eta > 0$. The constant ρ may be modified by a smaller one if necessary.

Using Theorem A.1.5 of [CC] again, for fixed δ as above, there is $\delta_1 = \delta_1(\delta, n) > 0$ such that $q_k \in \mathcal{WR}_{\delta_1, \frac{(\Lambda^2 + 1)\rho}{1 - \delta}}$ implies $q'_k \in \mathcal{R}_{\delta, \rho}$ for any $q'_k \in B_{g(t)}(q_k, \Lambda^2\rho)$. So it reduces to show $q_k \in \mathcal{WR}_{\delta_1, \frac{(\Lambda^2 + 1)\rho}{1 - \delta}}$ with respect to each time $t \in [t_k - \eta, t_k + \eta]$ for some $\eta > 0$ small enough. In fact, as showed in [Se], $d_{GH}(B_{g(t_k)}(q_k, \rho_1), B(\rho_1)) < \frac{1}{2}\delta_1\rho_1$ for some small number ρ_1 and all k large enough. By the boundedness of Ricci tensor again, there is a constant $\eta \le 1$ such that for each time $t \in [-\eta, \eta]$, we have $d_{GH}(B_{g(t_k + t)}(q_k, \rho_1), B_{g(t_k)}(q_k, \rho_1)) < \frac{1}{2}\delta_1\rho_1$ for all k. Thus $d_{GH}(B_{g(t_k + t)}(q_k, \rho_1), B(\rho_1)) < \delta_1\rho_1$ for each $t \in [-\eta, \eta]$. Now the result follows by setting $\rho = \frac{(1 - \delta)\rho_1}{\Lambda^2 + 1}$. $\square$

Note that in the proof, the constant $\delta_1 = \delta_1(\epsilon, n)$, so the constant η depends only on ϵ, n and $\bar{C}$. By assumption, there is a compact exhaustion $\{K_i\}_{i=1}^\infty$ of $\mathcal{R}$ and a sequence of smooth embeddings $F_i : K_i \to M$ such that $F_i(p_\infty) = p_i$ and $F_i^* g(t_i)$ converges to g_∞ in the local $C^{1,\alpha}$ sense. Following the lines described in [Se], we can prove

Lemma 4.5. *Denote by $K_{i,k} = F_k(K_i)$, then for any $\epsilon > 0$ and i, there are $k_0, \eta, \rho > 0$ such that*

$$\mathrm{Vol}(B_{g(t_k + t)}(q'_k, r)) \ge (1 - \epsilon)\mathrm{Vol}(B(r)), \forall q'_k \in K_{i,k}, k_0 < k, t \in [-\eta, \eta] \text{ and } r < \rho.$$

Now we are ready to prove the Proposition 4.1.

Proof of Proposition 4.1. Assume that $p_\infty \in K_i$ for each i. Set $\epsilon = \delta_0$ in the the previous lemma, where δ_0 is just the constant in Theorem 4.2, then for one fixed K_i, there exist $k_0, \eta, \rho > 0$ such that $\mathrm{Vol}(B_{g(t_k+t)}(q,r)) \geq (1-\delta_0)\mathrm{Vol}(B(r))$ whenever $q \in K_{i,k}, k_0 < k, t \in [-\eta, \eta]$ and $r < \rho$. Modifying ρ and η by smaller constants, we assume $(\epsilon_0 \rho)^2 \leq 2\eta < \tau$, where τ and ϵ_0 are constants in Lemma 4.3 and Theorem 4.2 respectively.

Let $h_k(\tilde{t})$ be the corresponding solutions to the unnormalized Ricci flow equation with initial value $h_k(0) = g(t_k - \eta)$, then $\mathrm{Vol}(B_{h_k(\tilde{t})}(q,r)) \geq (1-\delta_0)\mathrm{Vol}(B(r))$ whenever $q \in K_{i,k}, r < \rho, k_0 < k$ and $\tilde{t}$ satisfying $t(\tilde{t}) \in [0, 2\eta]$, since the inequality $\mathrm{Vol}(B(q,r)) \geq (1-\delta_0)\mathrm{Vol}(B(r))$ is scale invariant and $B_{h_k(\tilde{t})} \subset B_{g(t_k+t(\tilde{t}))}(q,r)$ for k large enough such that $t_k \geq T + \eta$ for T chosen as above. Denote by $\widetilde{Rm}_k$ the Riemannian curvature tensor of h_k, then by Theorem 4.2 and Lemma 4.3, we have

$$|Rm|(q, t_k + t) \leq C|\widetilde{Rm}_k|(q, \tilde{t}) \leq C(\tilde{t})^{-1} \leq C(t - t_k + \eta)^{-1},$$

for all $q \in K_{i,k}$. Hence $|Rm|(q,t)$ is uniformly bounded on $K_{i,k} \times [t_k - \frac{\eta}{2}, t_k + \frac{\eta}{2}]$.

By Hamilton's compactness theorem of Ricci flow solution, $\{(K_{i,k}, g(t_k+t), p_k)\}_{k=1}^\infty$ converge along a subsequence to a solution to the normalized Ricci flow $(K_{i,\infty}, g_{i,\infty}(t), p_{i,\infty}), t \in (-\frac{\eta}{2}, \frac{\eta}{2})$, in the local C^∞ sense. When we consider the time $t = 0$, then using a diagonalization argument, a subsequence of $\{(K_{i,k}, g(t_k), p_k)\}_{i,k}$ will converge in the local C^∞ sense to a smooth Riemannian manifold $(K_\infty, g_\infty, p_\infty)$, which is just $(\mathcal{R}, g_\infty)$, by the uniqueness of the limit space.

For fixed i, there is a family of metrics $g_{i,\infty}(t), t \in (-\frac{\eta}{2}, \frac{\eta}{2})$, on K_i. As showed in [Se], we translate the time by $\frac{\eta}{4}$, say considering the sequence $\{(K_{i,k}, g(t_k + \frac{\eta}{4} + t), p_k)\}_k$, and repeat the above argument, then obtain that $\{(K_{i,k}, g(t_k + t), p_k)\}_k \xrightarrow{C_{loc}^\infty} (K_{i,\infty}, g_{i,\infty}(t), p_{i,\infty})$ along another subsequence, on the time interval $t \in (-\frac{\eta}{2}, \frac{\eta}{4} + \frac{\eta}{2})$. The essential point is that the estimate $d_{GH}(B_{g(t_k)}(q_k, \rho_1), B(\rho_1)) < \frac{1}{2}\delta_1 \rho_1$ in the proof of Lemma 4.4 holds for some constant ρ_1, simultaneously the time t_k is replaced by $t_k + \frac{\eta}{4}$, but the constant η in Lemma 4.5 is fixed in this procedure. Iterating this process infinite times we obtain the convergence on K_i for all $t \in [0, \infty)$. Then do the same thing for each $K_i, i = 1, 2, \cdots$, and after a diagonalization argument, we get that a subsequence of $\{(K_{i,k}, g(t_k + t), p_k)\}_k$, say $(K_{i,k_i}, g(t_{k_i} + t), p_{k_i}) \xrightarrow{C_{loc}^\infty} (\mathcal{R}, g_\infty(t), p_\infty)$ for all $t \in [0, \infty)$, with $g_\infty(0) = g_\infty$.

We finally show that the completion of $\mathcal{R}$ with respect to the metric $g_\infty(t)$, say $\bar{\mathcal{R}}_t$, is just N_∞, for each time $t \in [0, \infty)$. Denote by $\mathcal{S} = N_\infty \backslash \mathcal{R}$ the set of singular points of $(N_\infty, g_\infty(0))$, then it suffice to show that $\bar{\mathcal{R}}_t = \mathcal{R} \cup \mathcal{S}$ for fixed time t. Assume $\mathcal{S} = \{q_l\}_{l=1}^Q$, where $Q \leq \beta$ for $\beta = \beta(M)$ by Lemma 3.5, and let $\varepsilon > 0$ be any small constant such that $B_{g_\infty(0)}(q_i, \varepsilon) \cap B_{g_\infty(0)}(q_j, \varepsilon) = \emptyset$ whenever $i \neq j$. Denote by $K_\varepsilon = \mathcal{R} \backslash \bigcup_{p_l} B_{g_\infty(0)}(p_l, \varepsilon)$, then using $|Ric_\infty| \leq \bar{C}$ on $\mathcal{R} \times [0, \infty)$ and by the evolution of the distance function, we obtain $d_{GH}((\mathcal{R} \backslash K_\varepsilon, g_\infty(t)), \mathcal{S}) \leq e^{2\bar{C}t}\varepsilon$ and consequently $\bar{\mathcal{R}}_t = \mathcal{R} \cup \mathcal{S}$, by letting $\varepsilon \to 0$. $\qquad\square$

5. Proofs of Theorems 1.1 and 1.2

The main result of this section is the following

Theorem 5.1. *Let $(M, \mathfrak{c})$ be a smooth oriented closed 4-manifold with a Spin^c-structure $\mathfrak{c}$. Assume that the first Chern class $c_1(\mathfrak{c})$ of $\mathfrak{c}$ is a monopole class of M satisfying that*

(5.1) $$c_1^2(\mathfrak{c})[M] \geq 2\chi(M) + 3\tau(M) > 0.$$

Let $g(t), t \in [0, \infty)$, be a solution to (1.3) so that $|Ric(g(t))| \leq 3$, and

(5.2) $$\lim_{t \to \infty} \overline{\lambda}_M(g(t)) = -\sqrt{32\pi^2 c_1^2(\mathfrak{c})[M]}.$$

Then there exists an $m \in \mathbb{N}$, and sequences of points $\{x_{j,k} \in M\}$, $j = 1, \cdots, m$, satisfying that, by passing to a subsequence,

$$(M, g(t_k + t), x_{1,k}, \cdots, x_{m,k}) \xrightarrow{d_{GH}} (\coprod_{j=1}^m N_j, g_\infty, x_{1,\infty}, \cdots, , x_{m,\infty}),$$

$t \in [0, \infty)$, in the m-pointed Gromov-Hausdorff sense for any $t_k \to \infty$, where (N_j, g_∞) $j = 1, \cdots, m$ are complete Kähler-Einstein orbifolds of complex dimension 2 with at most finitely many isolated orbifold points $\{q_i\}$. The scalar curvature (resp. volume) of g_∞ is

$$-Vol_{g_0}(M)^{-\frac{1}{2}}\sqrt{32\pi^2 c_1^2(\mathfrak{c})[M]} \quad (resp. \quad V = Vol_{g_0}(M) = \sum_{j=1}^m Vol_{g_\infty}(N_j)).$$

Furthermore, in the regular part of N_j, $\{g(t_k + t)\}$ converges to g_∞ in C^∞-sense.

Comparing with Proposition 3.2, Theorem 5.1 shows that the Einstein orbifolds are actually Kähler Einstein orbifolds under the additional assumptions. The key point in the proof is that the sequence of the self-dual parts of the curvatures of the connections on the determinant line bundles given by the irreducible solutions in the Seiberg-Witten equations converges to a non-trivial parallel self-dual 2-form on every component N_j, which is a candidate of the Kähler form.

Let $(M, \mathfrak{c})$ and $g(t)$ be the same as in Thoerem 5.1, and let V, m, t_k, $x_{j,k}$, $\check{R}(g(t))$, g_k, g_∞, N_j and $F_{j,k,r}$ be the same as in Section 3. Assume that, for each k, (ϕ_k, A_k) is an irreducible solution to the Seiberg-Witten equations (2.1). Let $|\cdot|_k$ denote the norm with respect to the metric $g_k = g(t_k)$. The following lemma shows that the L^2-norms of the self-dual parts $F_{A_k}^+$ tends to zero.

Lemma 5.2.
$$\lim_{k \to \infty} \int_M |\nabla^k F_{A_k}^+|_k^2 dv_k = 0,$$

where ∇^k is the connection on $\Lambda^2 T^*(M)$ induced by Levi-civita connection.

Proof. The Bochner formula implies that

$$0 = -\frac{1}{2}\Delta_k|\phi_k|_k^2 + |\nabla^{A_k}\phi_k|_k^2 + \frac{R(g_k)}{4}|\phi_k|_k^2 + \frac{1}{4}|\phi_k|_k^4,$$

By taking integration we get that

(5.3) $$\int_M (|\nabla^{A_k}\phi_k|_k^2 + \frac{R(g_k)}{4}|\phi_k|_k^2)dv_k = -\frac{1}{4}\int_M |\phi_k|_k^4 dv_k.$$

Since $\lambda_M(g_k)$ is the lowest eigenvalue of the operator $-4\triangle_k + R(g_k)$, for any $1 \gg \epsilon > 0$, by definition

$$(5.4) \qquad \lambda_M(g_k)\int_M |\phi_k|^2_{k,\epsilon}dv_k \leq \int_M (4|\nabla|\phi_k|_{k,\epsilon}|^2 + R(g_k)|\phi_k|^2_{k,\epsilon})dv_k,$$

where $|\cdot|^2_{k,\epsilon} = |\cdot|^2_k + \epsilon^2$. By Kato's inequality (cf. (2.5)) and letting $\epsilon \to 0$,

$$\lambda_M(g_k)\int_M |\phi_k|^2_k dv_k \leq \int_M (4|\nabla^{A_k}\phi_k|^2_k + R(g_k)|\phi_k|^2_k)dv_k = -\int_M |\phi_k|^4_k dv_k \leq 0.$$

As $\lambda_M(g_k) \leq 0$, by Schwarz inequality,

$$\overline{\lambda}_M(g_k)(\int_M |\phi_k|^4_{k,\epsilon}dv_k)^{\frac{1}{2}} = \lambda_M(g_k)\mathrm{Vol}_{g_k}(M)^{\frac{1}{2}}(\int_M |\phi_k|^4_{k,\epsilon}dv_k)^{\frac{1}{2}} \leq \lambda_M(g_k)\int_M |\phi_k|^2_{k,\epsilon}dv_k.$$

Therefore

$$\overline{\lambda}_M(g_k)(\int_M |\phi_k|^4_{k,\epsilon}dv_k)^{\frac{1}{2}} \leq \int_M (4|\nabla|\phi_k|_{k,\epsilon}|^2 + R(g_k)|\phi|^2_{k,\epsilon})dv_k.$$

Thus

$$(5.5) \qquad 4\int_M (|\nabla^{A_k}\phi_k|^2_k - |\nabla|\phi_k|_{k,\epsilon}|^2)dv_k \leq -\int_M |\phi_k|^4_k dv_k - \overline{\lambda}_M(g_k)(\int_M |\phi_k|^4_{k,\epsilon}dv_k)^{\frac{1}{2}}.$$

From (2.5), $|\nabla|\phi_k|_{k,\epsilon}|^2 \leq \frac{3}{4}|\nabla^{A_k}\phi_k|^2_k$. Hence, by letting $\epsilon \longrightarrow 0$, we have

$$(5.6) \qquad \int_M |\nabla^{A_k}\phi_k|^2_k dv_k \leq -((\int_M |\phi_k|^4_k dv_k)^{\frac{1}{2}} + \overline{\lambda}_M(g_k))(\int_M |\phi_k|^4_k dv_k)^{\frac{1}{2}}.$$

If $c^+_{1,k}$ denotes the self-dual part of the harmonic form representing the first Chern class $c_1(\mathfrak{c})$ of $\mathfrak{c}$, by the Seiberg-Witten equation we get that

$$(5.7) \qquad \int_M |\phi_k|^4_k dv_k = 8\int_M |F^+_{A_k}|^2 dv_k \geq 32\pi^2 [c^+_{1,k}]^2[M] \geq 32\pi^2 c^2_1(\mathfrak{c})[M].$$

Note that, by the standard estimates for Seiberg-Witten equations,

$$-\breve{R}(g_k) \geq |\phi_k|^2_k$$

and, by Theorem 1.1 in [FZ], $\sqrt{32\pi^2 c^2_1(\mathfrak{c})[M]} + \overline{\lambda}_M(g_k)$ is non-positive. Hence (5.8)

$$\begin{aligned}\int_M |\nabla^{A_k}\phi_k|^2_k dv_k &\leq -(\sqrt{32\pi^2 c^2_1(\mathfrak{c})[M]} + \overline{\lambda}_M(g_k))(\int_M |\phi_k|^4_k dv_k)^{\frac{1}{2}} \\ &\leq \breve{R}(g_k)V^{\frac{1}{2}}(\sqrt{32\pi^2 c^2_1(\mathfrak{c})[M]} + \overline{\lambda}_M(g_k)) \longrightarrow 0,\end{aligned}$$

when $k \longrightarrow \infty$, by (5.2) and Lemma 3.1.

By the second one of the Seiberg-Witten equations again (cf. [Le2]),

$$(5.9) \qquad |\nabla^k F^+_{A_k}|^2_k \leq \frac{1}{2}|\phi_k|^2_k|\nabla^{A_k}\phi_k|^2_k,$$

where ∇^{A_k} is the connection on $\Gamma(S_{\mathfrak{c}})$ induced by the Levi-civita connection. Hence

$$\int_M |\nabla^k F^+_{A_k}|^2_k dv_k \leq \frac{1}{2}|\breve{R}(g(t_k))|\int_M |\nabla^{A_k}\phi_k|^2_k dv_k \longrightarrow 0,$$

when $k \longrightarrow \infty$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Regard $F_{A_k}^+$ as self-dual 2-forms of g_k' on $U_{j,r} = B_{g_\infty}(x_{j,\infty}, r) \backslash \bigcup_i B_{g_\infty}(q_{i,j}, r^{-1})$, where $g_k' = F_{j,k,r+1}^* g_k$, and $q_{i,j}$ are the orbifold points of N_j. Since

$$(5.10) \qquad |F_{A_k}^+|_k^2 = \frac{1}{8}|\phi_k|_k^4 \le \frac{1}{8}\breve{R}(g_k)^2 \le C,$$

where C is a constant independent of k, $F_{A_k}^+ \in L^{1,2}(g_k')$, and

$$\|F_{A_k}^+\|_{L^{1,2}(g_k')} \le C',$$

where C' is a constant independent of k. Note that $\| \cdot \|_{L^{1,2}(g_\infty)} \le 2\| \cdot \|_{L^{1,2}(g_k')}$ for $k \gg 1$ since $g_k' \xrightarrow{C^{1,\alpha}} g_\infty$ on $U_{j,r}$. Thus, by passing to a subsequence, $F_{A_k}^+ \xrightarrow{L^{1,2}} \Omega_j \in L^{1,2}(g_\infty)$, a self-dual 2-form with respect to g_∞.

Lemma 5.3. *For any j, Ω_j is a smooth self-dual 2-form on $U_{j,r} \backslash \partial U_{j,r}$ such that $\nabla^\infty \Omega_j \equiv 0$, and $|\Omega_j|_\infty \equiv \text{cont.} \ne 0$, where ∇^∞ is the connection induced by the Levi-civita connection of g_∞. Hence, g_∞ is a Kähler metric with Kähler form $\sqrt{2}\frac{\Omega_j}{|\Omega_j|}$ on $U_{j,r}$.*

Proof. By Lemma 5.2

$$0 \le \int_{U_{j,r}} |\nabla^\infty \Omega_j|_\infty^2 dv_\infty = \lim_{k \to \infty} \int_{U_{j,r}} |\nabla^\infty F_{A_k}^+|_\infty^2 dv_\infty \le 2 \lim_{k \to \infty} \int_M |\nabla^k F_{A_k}^+|_k^2 dv_k = 0.$$

It is easy to see that Ω_j is a weak solution of the elliptic equation $\nabla^\infty \Omega_j = 0$ on $U_{j,r}$. By elliptic equation theory, Ω_j is a smooth self-dual 2-form on $U_{j,r} \backslash \partial U_{j,r}$, $\nabla^\infty \Omega_j \equiv 0$, and $|\Omega_j|_\infty \equiv \text{cont.}$.

Now we claim that, for any j and $r \gg 1$, $\int_{U_{j,r}} |\Omega_j|_\infty^2 dv_\infty \ne 0$. If not, there exist j_s, $s = 1, \cdots, m_0$, $m_0 \le m$, such that $\int_{U_{j_s,r}} |\Omega_{j_s}|_\infty^2 dv_\infty \equiv 0$. By Lemma 3.1, $\overline{R}_\infty = \lim_{k \to \infty} \overline{R}(g_k) = \lim_{k \to \infty} \breve{R}(g_k) = \overline{\lambda}_M V^{-\frac{1}{2}}$, which is the scalar curvature of g_∞, i.e. $\overline{R}_\infty = R(g_\infty)$. Note that, by (5.10) and Lemma 3.7,

$$\begin{aligned}
\int_{U_{j,r}} |\Omega_j|_\infty^2 dv_\infty &= \lim_{k \to \infty} \int_{U_{j,r}} |F_{A_k}^+|_k^2 dv_k \\
&\le \frac{1}{8} \lim_{k \to \infty} \breve{R}(g_k)^2 \text{Vol}_{g_k'}(U_{j,r}) \\
&= \frac{1}{8}\overline{R}_\infty^2 \text{Vol}_{g_\infty}(U_{j,r}),
\end{aligned}$$

$$\begin{aligned}
\lim_{k \to \infty} |\int_M |F_{A_k}^+|_k^2 dv_k - \sum_{j=1}^m \int_{U_{j,r}} |F_{A_k}^+|_k^2 dv_k| &\le \frac{1}{8} \lim_{k \to \infty} \breve{R}(g_k)^2 \text{Vol}_{g_k}(M \backslash \bigcup_j F_{k,j,r}(U_{j,r})) \\
&\le \frac{1}{8}C\overline{R}_\infty^2 \sum_{j=1}^m \text{Vol}_{g_\infty}(N_j \backslash U_{j,\frac{r}{2}}),
\end{aligned}$$

and, by Lemma 3.1,

$$\lim_{k \to \infty} |\int_M (R(g_k)^2 - \overline{R}_\infty^2) dv_k| \le 24 \lim_{k \to \infty} \int_M (|R(g_k) - \overline{R}(g_k)| + |\overline{R}_\infty - \overline{R}(g_k)|) dv_k = 0,$$

where C is a constant in-dependent of k. Hence, we obtain

$$
\begin{aligned}
\overline{R}_\infty^2 \sum_{j \neq j_1, \cdots, j_{m_0}} \mathrm{Vol}_{g_\infty}(\mathrm{U_{j,r}}) \ &\geq\ \sum_{j=1}^m \int_{U_{j,r}} 8|\Omega_j|_\infty^2 dv_\infty = \lim_{k \longrightarrow \infty} \sum_{j=1}^m \int_{U_{j,r}} 8|F_{A_k}^+|_k^2 dv_k \\
&\geq\ \lim_{k \longrightarrow \infty} \int_M 8|F_{A_k}^+|_k^2 dv_k - C\overline{R}_\infty^2 \sum_{j=1}^m \mathrm{Vol}_{g_\infty}(\mathrm{N_j} \backslash \mathrm{U}_{\mathrm{j},\frac{r}{2}}) \\
&\geq\ 32\pi^2 c_1^2(\mathfrak{c})[M] - C\overline{R}_\infty^2 \sum_{j=1}^m \mathrm{Vol}_{g_\infty}(\mathrm{N_j} \backslash \mathrm{U}_{\mathrm{j},\frac{r}{2}}).
\end{aligned}
$$

The last inequality is obtained by (5.7). Thus, by (5.1),

$$
\overline{R}_\infty^2 \sum_{j \neq j_1, \cdots, j_{m_0}} \mathrm{Vol}_{g_\infty}(\mathrm{U_{j,r}}) \geq 32\pi^2(2\chi(\mathrm{M}) + 3\tau(\mathrm{M})) - C\overline{R}_\infty^2 \sum_{j=1}^m \mathrm{Vol}_{g_\infty}(\mathrm{N_j} \backslash \mathrm{U}_{\mathrm{j},\frac{r}{2}}).
$$

By the Chern-Gauss-Bonnet formula and the Hirzebruch signature theorem,

$$
2\chi(M) + 3\tau(M) \geq \frac{1}{4\pi^2} \int_{U_{k,r}} (\frac{1}{24}R(g_k)^2 + 2|W^+(g_k)|_k^2)dv_k - \frac{1}{8\pi^2} \int_M |Ric^\circ(g_k)|^2 dv_k.
$$

By Lemma 3.1, and the fact that $g_k' \xrightarrow{L^{2,p}} g_\infty$ on $U_{j,r}$, we obtain that

$$
\begin{aligned}
\overline{R}_\infty^2 \sum_{j \neq j_1, \cdots, j_{m_0}} \mathrm{Vol}_{g_\infty}(\mathrm{U_{j,r}}) \ &\geq\ \sum_{j=1}^m 8 \int_{U_{j,r}} (\frac{\overline{R}_\infty^2}{24} + 2|W^+(g_\infty)|_\infty^2)dv_\infty \\
&\quad - C\overline{R}_\infty^2 \sum_{j=1}^m \mathrm{Vol}_{g_\infty}(\mathrm{N_j} \backslash \mathrm{U}_{\mathrm{j},\frac{r}{2}}).
\end{aligned}
$$

Note that, on any $U_{j,r}$, $j \neq j_1, \cdots, j_{m_0}$, $\nabla^\infty \Omega_j \equiv 0$, $|\Omega_j|_\infty \equiv$ cont. $\neq 0$, and Ω_j is a self-dual 2-form. Thus g_∞ is a Kähler metric with Kähler form $\sqrt{2}\frac{\Omega_j}{|\Omega_j|}$ on $U_{j,r}$, $j \neq j_1, \cdots, j_{m_0}$. It is well known that $\overline{R}_\infty^2 = 24|W^+(g_\infty)|_\infty^2$ for Kähler metrics (cf. [Le3]). Thus

$$
\begin{aligned}
\overline{R}_\infty^2 \sum_{j \neq j_1, \cdots, j_{m_0}} \mathrm{Vol}_{g_\infty}(\mathrm{U_{j,r}}) \ &\geq\ \overline{R}_\infty^2 \sum_{j \neq j_1, \cdots, j_{m_0}} \mathrm{Vol}_{g_\infty}(\mathrm{U_{j,r}}) - C\overline{R}_\infty^2 \sum_{j=1}^m \mathrm{Vol}_{g_\infty}(\mathrm{N_j} \backslash \mathrm{U}_{\mathrm{j},\frac{r}{2}}) \\
&\quad + \sum_{j_s = j_1, \cdots, j_{m_0}} 8 \int_{U_{j,r}} (\frac{\overline{R}_\infty^2}{24} + 2|W^+(g_\infty)|_\infty^2)dv_\infty \\
&\geq\ \overline{R}_\infty^2 \sum_{j \neq j_1, \cdots, j_{m_0}} \mathrm{Vol}_{g_\infty}(\mathrm{U_{j,r}}) - C\overline{R}_\infty^2 \sum_{j=1}^m \mathrm{Vol}_{g_\infty}(\mathrm{N_j} \backslash \mathrm{U}_{\mathrm{j},\frac{r}{2}}) \\
&\quad + \frac{1}{3} \sum_{j_s = j_1, \cdots, j_{m_0}} \overline{R}_\infty^2 \mathrm{Vol}_{g_\infty}(\mathrm{U_{js,r}}).
\end{aligned}
$$

Note that, for $r \gg 1$,

$$1 \gg 3C\overline{R}_\infty^2 \sum_{j=1}^{m} \mathrm{Vol}_{g_\infty}(N_j \backslash U_{j,\frac{r}{2}}) \geq \sum_{j_s = j_1, \cdots, j_{m_0}} \overline{R}_\infty^2 \mathrm{Vol}_{g_\infty}(U_{j_s, r}).$$

A contradiction. Thus, for all j, $\int_{U_{j,r}} |\Omega_j|_\infty^2 dv_\infty \neq 0$, and $\nabla^\infty \Omega_j \equiv 0$, $|\Omega_j|_\infty \equiv \mathrm{cont.} \neq 0$. Thus we obtain the conclusion.

$\square$

Proof of Theorem 5.1. First, assume that $diam_{g(t_k)}(M) \longrightarrow \infty$, when $k \longrightarrow \infty$. By Proposition 3.2 and Proposition 4.1, there exists a $m \in \mathbb{N}$, and a sequence of points $\{x_{j,k} \in M\}$, $k \in \mathbb{N}$, $j = 1, \cdots, m$, satisfying that, by passing to a subsequence, $(M, g(t_k+t), x_{1,k}, \cdots, x_{m,k})$, $t \in [0, \infty)$, converges to $\{(N_1, g_\infty, x_{1,\infty}), \cdots, (N_m, g_\infty, x_{m,\infty})\}$ in the m-pointed Gromov-Hausdorff sense, when $k \longrightarrow \infty$, where (N_j, g_∞) $j = 1, \cdots, m$ are complete Einstein 4-orbifolds with finite isolated orbifold points $\{q_i\}$. The scalar curvature of g_∞ is

$$\overline{R}_\infty = \lim_{t \longrightarrow \infty} \lambda_M(g(t)), \qquad \text{and} \qquad V = \mathrm{Vol}_{g_0}(M) = \sum_{j=1}^{m} \mathrm{Vol}_{g_\infty}(N_j).$$

By Lemma 5.2, g_∞ is a Kähler-Einstein metric in the non-singular part of $\coprod_{j=1}^{m} N_j$. Then by the same arguments as in Section 4 of [Ti], g_∞ is actually a Kähler-Einstein orbifold metric. Furthermore, in the non-singular part of $\coprod_{j=1}^{m} N_j$, $\{g(t_k+t)\}$, $t \in [0, \infty)$, C^∞-converges to g_∞ by Proposition 4.1.

If $diam_{g_k}(M) < C$ for a constant C in-dependent of k, we can also obtain the conclusion by the similar, but much easier, arguments as above.

$\square$

Theorem 5.4. *Let $(M, \mathfrak{c})$ be a smooth compact closed oriented 4-manifold with a Spin^c-structure $\mathfrak{c}$. Assume that the first Chern class $c_1(\mathfrak{c})$ of $\mathfrak{c}$ is a monopole class of M satisfying $c_1^2(\mathfrak{c})[M] = 2\chi(M) + 3\tau(M) > 0$, and $\chi(M) = 3\tau(M)$. If M admits a solution $g(t), t \in [0, \infty)$ to (1.3) with $|R(g(t))| \leq 12$, then*

$$\lim_{t \longrightarrow \infty} \overline{\lambda}_M(g(t)) = -\sqrt{32\pi^2 c_1^2(\mathfrak{c})[M]}.$$

Furthermore, if $|Ric(g(t))| \leq 3$, the Kähler-Einstein metric g_∞ in Theorem 5.1 is a complex hyperbolic metric.

Proof. Let $V = Vol_{g(t)}(M)$. By the Chern-Gauss-Bonnet formula and the Hirzebruch signature theorem,

$$(5.11) \quad 2\chi(M) - 3\tau(M) \geq \frac{1}{4\pi^2} \int_M (\frac{1}{24}R(g(t))^2 + 2|W^-(g(t))|^2 - \frac{1}{2}|Ric^o(g(t))|^2 dv_{g(t)},$$

where W^- is the anti-self-dual Weyl tensor. Note that

$$(5.12) \quad \int_M R(g(t))^2 dv_{g(t)} \geq \overline{R}(g(t))^2 V \longrightarrow \overline{R}_\infty^2 V = \lim_{t \longrightarrow \infty} \overline{\lambda}_M(g(t))^2,$$

when $t \longrightarrow \infty$, by Schwarz inequality and Lemma 3.1. By (5.11), (5.12), Lemma 3.1 and Theorem 1.1 in [FZ],

$$
\begin{aligned}
2\chi(M) - 3\tau(M) &\geq \liminf_{t \longrightarrow \infty} \frac{1}{2\pi^2} \int_M |W^-(g(t))|^2 dv_{g(t)} + \frac{1}{96\pi^2} \lim_{t \longrightarrow \infty} \overline{\lambda}_M(g(t))^2 \\
&\geq \liminf_{t \longrightarrow \infty} \frac{1}{2\pi^2} \int_M |W^-(g(t))|^2 dv_{g(t)} + \frac{1}{3} c_1^2(\mathfrak{c})[M] \\
&= \liminf_{t \longrightarrow \infty} \frac{1}{2\pi^2} \int_M |W^-(g(t))|^2 dv_{g(t)} + \frac{1}{3}(2\chi(M) + 3\tau(M)).
\end{aligned}
$$

Since $\chi(M) = 3\tau(M)$, we obtain

$$
\lim_{t \longrightarrow \infty} \overline{\lambda}_M(g(t)) = -\sqrt{32\pi^2 c_1^2(\mathfrak{c})[M]},
$$

and

$$
\liminf_{t \longrightarrow \infty} \frac{1}{2\pi^2} \int_M |W^-(g(t))|^2 dv_{g(t)} = 0.
$$

Now, assume that $|Ric(g(t))| \leq 3$. Let t_k, N_j, g_k, and g_∞ be the same as above. For any j and compact subset U of the regular part of N_j,

$$
0 \leq \int_U |W^-(g_\infty)|_\infty^2 dv_\infty \leq \liminf_{k \longrightarrow \infty} \int_M |W^-(g(t_k))|_k^2 dv_k = 0,
$$

since $g(t_k) \xrightarrow{L^{2,p}} g_\infty$ on U. Hence g_∞ is a Kähler-Einstein metric with $W^-(g_\infty) \equiv 0$. This implies that g_∞ is a complex hyperbolic metric (cf. [Le1]). The desired result follows.

$\square$

Proofs of Theorem 1.1 and Theorem 1.2. By the work of Taubes [Ta], if (M, ω) is a compact symplectic manifold with $b_2^+(M) > 1$, the spin^c-structure induced by ω is a monopole class. Moreover, since in this situation $c_1^2(\mathfrak{c})[M] = 2\chi(M) + 3\tau(M)$, Theorem 1.1 (resp. Theorem 1.2) is an obvious consequence of Theorem 5.1 (resp. Theorem 5.4). $\square$

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